Free Knots, Groups, and Finite-Type Invariants

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Abstract

Based on a recently introduced by the author notion of *parity*, in the present paper we construct a sequence of invariants (indexed by natural numbers m) of long virtual knots, valued in certain simply-defined group $\tilde{G}_m$ (the Cayley graphs of these groups are represented by grids in the $(m + 1)$-space); the conjugacy classes of elements of G_m play the role of invariants of *compact* virtual knots. By construction, all invariants do not change under *virtualization*. Factoring the group algebra of the corresponding group by certain polynomial relations leads to finite order invariants of (long) knots which do not change under *virtualization*.

1   Introduction

Virtual knot theory was invented by Kauffman, [Ka1]; virtual knots correspond to knots in thickened surfaces $S_g \times S^1$, considered up to isotopies and stabilizations.

The theory of finite type invariants of classical knots was invented independently by V.A.Vassiliev [Vas] and M.N.Goussarov [Gou], and it turned out that many well-known invariants are expressible in terms of finite-type invariants [BL, BN]; a breakthrough in Vassiliev's theory of finite type invariants was marked by the celebrated paper by Kontsevich [Kon], where the structure of the space of finite-type Vassiliev invariant was explicitly described. The Vassiliev knot invariants initiated the study of virtual knots by Goussarov, Polyak, and Viro [GPV]. In the latter paper, the authors gave a definition of virtual knots equivalent to the original one due to Kauffman; this allowed one to classify invariants of finite type of *classical and virtual knots* in terms of combinatorial formulae, though their definition of finite type invariants of virtual knots was quite limited. In the present paper, we shall use a more general (and more natural) definition of finite-type (Vassiliev) invariants of virtual knots due to Kauffman [Ka1].

In the case of knots in thickened surfaces (without stabilization), an analogue of the Kontsevich theorem was obtained in [AM]; however this solution of the classification problem (an explicit universal formula) does not look as elegant as in the classical case; in any case, this formula is quite far away from practical implementation.

Virtual knots are much more complicated objects than classical ones: all invariants of classical knots of order zero are constants, whence the space of finite type invariants of order zero is infinite-dimensional. The latter statement can be reformulated in the following manner: there are infinitely many types of virtual knots where two virtual knots represent the same type whenever one of them

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can be obtained from the other by a sequence of (equivalences and) crossing changes. These "types" (or "equivalence classes") are called *flat virtual knots* (see [HK]); representing themselves the dual space of the space of invariants of order 0 they are important for investigation of invariants of finite type.

A thorough simplification for the notion of flat virtual knot is the notion of *free knot*. In [Tu], V.G.Turaev (who first introduced free knots under the name of "homotopy classes of Gauss words") conjectured that all free virtual knots were trivial; this conjecture was first disproved by the author in [Ma], and later, in [Gib].

The aim of the present work is to construct invariants of free knots valued in certain groups G_m (depending on a certain natural parameter m), see [MM, MM2], and to extend this construction to some simply defined infinite series of finite type invariants for (long) virtual knots. The groups G_m are defined by generators and relators, and have very simple Cayley graphs; the group G_1 is isomorphic to the infinite dihedral group.

I am very grateful to O.V.Manturov for fruitful consultations.

2   Basic Definitions

Throughout the paper, by *graph* we mean a finite multi-graph (loops and multiple edges are allowed). Here a 4-valent graph is called *framed* if for each vertex of it, the four emanating half-edges are split into two pairs; half-edges belonging to the same pair are called (formally) *opposite*. Assume a framed 4-valent graph K is given by its Gaussian diagram $C(K)$. We shall use the generic term "4-graph" to denote a topological object obtained from a four-valent graph by adding free circles (without vertices) as connected components.

For framed 4-graphs one naturally defines unicursal components: if a graph has connected components homeomeorphic to the circle, they are treated as unicursal components; for the rest of the graph, unicursal components are defined as equivalence classes of edges generated by "local opposite" relation at vertices. This relation of unicursal components naturally yields the *number of uncursal components* which agrees with the number of components of a link diagram drawn on the plane. Certainly, a framed 4-graph can be encoded by a chord diagram if and only if it can be has one unicursal component.

Definition 2.1. By a *chord diagram* we mean a pair $(S_0 \sqcup \cdots \sqcup S_0 \subset S^1)$ consisting of an oriented circle (called *the core circle)* and a set of n unordered pairs of points (all $2n$ points are pairwise distinct). These pairs are called *chords*; points of the pairs are called *ends of chords*.

In figures, we shall depict chords by solid lines connecting pairs of points on the core circle. We say that a chord diagram is *labeled* if it is marked by a point on the core circle distinct from chord ends. We consider chord diagrams up to a natural equivalence (orientation-preserving homeomorphism of core circle which respects the collection of chords).

Framed 4-graphs with one unicursal component are in one-one correspondence with chord diagrams. The 4-graph without vertices corresponds to the chord diagram without chords. Every framed
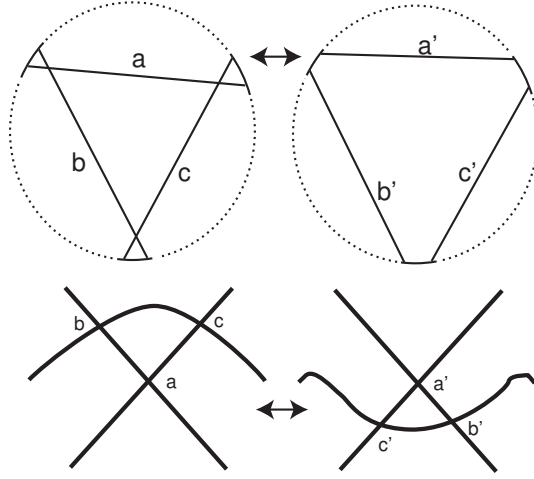
Figure 1: The third Reidemeister move and the correspondence between crossings

4-valent graph can be represented as the image of the circle going transversely through all edges. The two preimages of every vertex correspond to a chord.

One can naturally consider *oriented* or *non-oriented* chord diagrams (framed 4-graphs).

All statements about chord diagrams can be translated into the language of framed 4-graphs and vice versa.

By a *free knot* [Ma] is meant an equivalence class of chord diagrams by the equivalence relation generated by the following three elementary equivalences. The first Reidemeister move corresponds to an addition/removal of a solitary chord. The second Reidemeister move is an addition/removal of a couple of *similar* chords. Here chords a and b are called similar if their chord ends a_1, a_2 and b_1, b_2 can be numbered in a way such that a_1 and b_1 are adjacent and a_2 and b_2 are adjacent (here chord ends x, y are *adjacent* if one of the two components of the complement $C \backslash \{x, y\}$ has no chord ends inside it).

The third Reidemeister move is depicted in Fig. 1.

Here the only changing part of the chord diagram consists of three segments with two chord ends on each. The chords belonging to the "stable" part of the chord diagram are not drawn not depicted, and the part of the circle containing only stable chord ends is depicted by dotted lines; in the remaining part, every chord "moves" each of its two ends from one position to the other.

All the three Reidemeister moves on chord diagram originate from the Reidemeister moves for virtual knots [Ka1], defined later in the present paper.

Every Reidemeister move transforms one fragment of the frame four-graph. When depicting such a fragment in a figure we show only the "changing part", leaving the rest of the diagram outside. In the case of one unicursal component this corresponds to a rule for transforming a Gauss diagram; this transformation deals with some arcs of the Gauss diagrams. When depicting on the plane, we do not show "stable chords"; we use dotted line for depicting "stable parts" of the core circle.

Definition 2.2. By a *long free knot* we mean an equivalence class of based chord diagrams by the
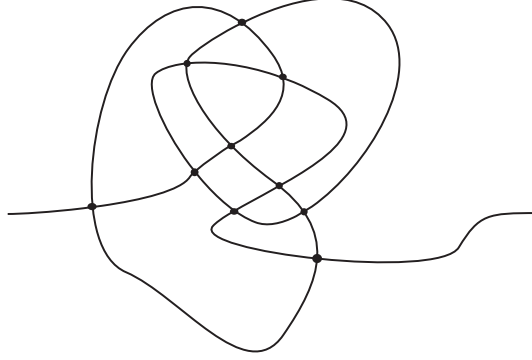
Figure 2: A long free knot

same Reidemeister move; here we only require that the marked point lies outside the transformation domain (equivalently, this marked point lies on a dotted part of the core circle), see Fig.2.

3 The Groups G_m. Even and Odd Chords

Fix a positive integer m. Consider the group $G_m = \langle a_0', a_0'', \ldots, a_{m-1}', a_{m-1}'', a_m | (a_m)^2 = e, (a_i')^2 = e, (a_i'')^2 = e, i = 0, \ldots, m-1, a_i' a_j' = a_j' a_i'', i < j; a_i' a_j'' = a_j'' a_i'', i < j; a_i' a_m = a_m a_i'' \rangle$.

With every framed chord diagram D we associate an element G_m. For this sake, we associate with each chord its *index and type* in the following way.

Let f be the operation on chord diagrams deleting all odd chords. It is proved in [Ma] that this operation is well defined: if two chord diagrams D, D' represent equivalent free knots, then so are $f(D), f(D')$.

We say that a chord diagram D a chord has index 0 if this chord *odd*; those *even chords* of D which become odd in $f(D)$, are decreed to have index 1, inductively, for $m-1$ we define those chords of D which remain in $f^{m-1}(D)$ and become odd ones in $f^{m-1}(D)$, to have index $m-1$; all the remaining chords are said to have index m.

Now, for chords of index m, we do not define the type; for chords of index $k < m$ we define the type (*the first type* or *the second type*) depending on the fact whether the number of chords of the diagram $f^k(D)$, linked with the given chord, is *even* (in $f^k(D)$) or odd.

Let us take a chord diagram D and let us start walking from the base point of the diagram D along the orientation of the circle. Whenever we meet a chord end of index i and type a, we write down the generator of the group G with index i and, if index is less than m, with the number of primes equal to a. If the index is equal to m then we write down the generator a_m. When we make a full turn returning to the base point, we get a certain word $\gamma(D)$. Denote the corresponding element of the group G by $[\gamma(D)]$, see Fig. 3.

In [MM, MM2], the following statements are proved

Theorem 1. *1.* $[\gamma(D)]$ *is an invariant of long free knots. 2. The conjugacy class of the element* $[\gamma(D)]$ *is an invariant of free knots.*
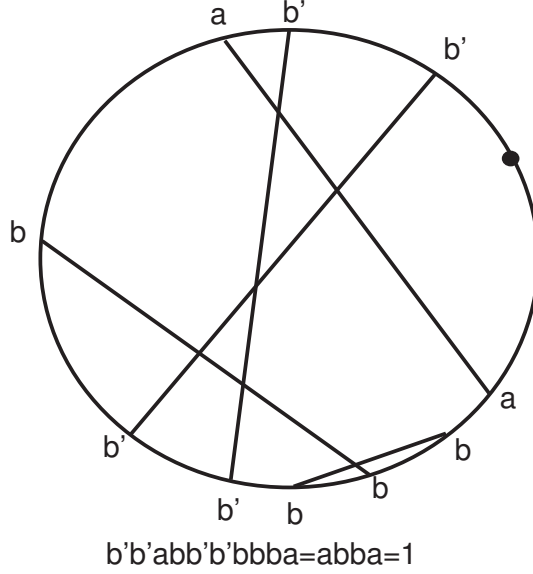
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b'b'abb'b'bbba=abba=1

Figure 3: Constructing $\gamma(D)$ from a chord diagram

The proof of this theorem follows from a straightforward check.

3.1 The Cayley graph of G_m

Elements of G_m are in one-one correspondence with integer points in the Euclidean space $\mathbf{R}^{m+1}$ with last coordinate equal to zero or 1. The unit of the group is represented by the origin of coordinates. With an element of the group, we associate a point in $\mathbf{R}^{m+1}$ defined by induction, as follows. Assume for some elements $g \in G_m$ represented by a word in generators, we have already defined the corresponding point in the Euclidean space; let us define those points corresponding to $g\alpha$, where α is the generator of G_m. The right multiplication by the generator with lower index k (a'_k or a''_k or a_k for $k = m$) correspond to the shift of the $(k+1)$-th coordinate. The direction of the shift is defined as follows: for $0 \leq k \leq m-1$ if the sum of coordinates of the initial point (all except the first k) is even then the multiplication by a'_k increases the first coordinate, whence the multiplication by a''_k decreases it; if this sum of all coordinates except the first k is odd, then a'_k and a''_k change their roles; the multiplication by a_m changes the last $(m+1)$-st coordinate: from zero to one and from one to zero. It can be readily checked that this correspondence is well defined, i.e., the resulting element of the group G_m does not depend on the way of representing an element of the group as a word in generators.

It can be easily checked that the last coordinated of the element of G_m corresponding to a word coming from a chord diagram, is zero. The cojugacy class of the element of G_m having coordinates $(x_0, \ldots, x_{m-1}) \in \mathbb{Z}^m$ (with the last coordinate equal to zero) consists of elements with coordinates $(\pm x_0, \ldots, \pm x_{m-1})$ having the last coordinate equal to zero.

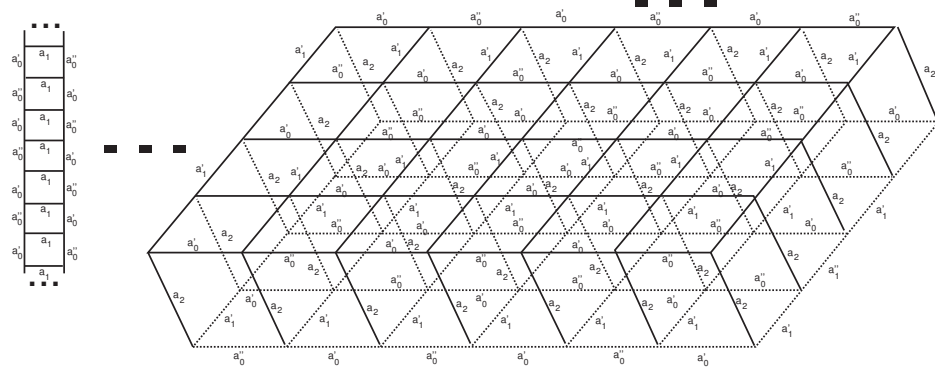The Cayley graphs of G_1 and G_2 are given in Fig. 4.

Figure 4: The Cayley graphs for G_1 and G_2

3.2   Virtual knots

In the present paper, virtual knots appear as a generalization of free knots with special decorations at vertices of the framed 4-graph (resp., chords of the chord diagram). We shall give a definition in terms of planar diagrams and Reidemeister moves, the corresponding moves on chord diagram with decorations (called *Gauss diagrams*) can be written straightforwardly.

A *virtual diagram*, an example see in Fig. 5, is a generic planar immersion of a four-valent graph, where every image of a vertex is marked as a classical crossing (one pair of opposite edges is said to form an overcrossing, and the other pair is said to form an undercrossing); when depicting on the plane, undercrossings are marked by a broken line; we also encircle the intersection points between images of edges and say that they from *virtual crossings*. We also admit the case when some components of the virtual diagram are not graphs, but circles without vertices. For this sake, we slightly change the notation and will say that a virtual diagram is a generic immersion of a 4-graph in $\mathbb{R}^2$, where a 4-graph means a disjoint sum of a 4-valent graph with a collection of circles.

Remark 3.1. Unlike a framed 4-graph, in a virtual diagram, we have got two new structures at classical crossings: some pair of opposite edges are said to form an *overcrossing*, the remaining pair forms an *undercrossing*; besides, the four edges incident to classical crossing obtain a *counterclockwise ordering*: being immersed in the plane, we know not only which edges are opposite, but also which non-opposite edge follows after the given one in the counterclockwise direction.

Definition 3.1. A *virtual link* is an equivalence class of virtual diagram modulo three classical Reidemeister moves and the detour move, see [Ma3]. The Reidemeister moves are applied to fragments of the diagram containing only classical crossings. The detour move deals with an arc containing only virtual crossings and virtual self-crossings. This move replaces such an arc with another (possibliy, self-intersecting) arc having the same endpoints; all new crossings and self-crossings are marked as virtual ones.

For a virtual diagram, one naturally defines *unicursal components*. The immersion image of the 4-
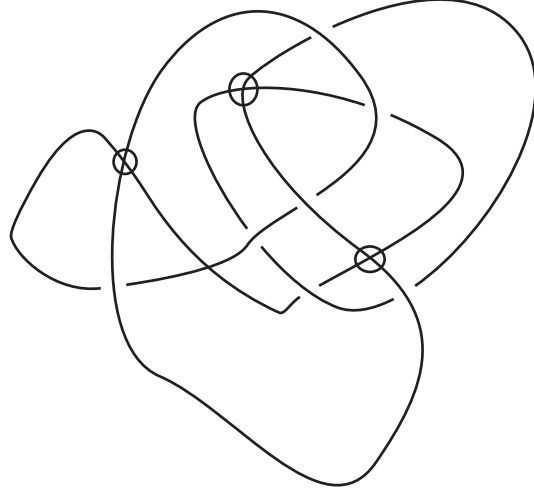
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Figure 5: A virtual diagram

graph is again a 4-graph; the latter consists of a 4-valent graph and a collection of circles disjoint from the rest of the graph. Every circle is treated as a unicursal component; other unicursal components are equivalence classes of edges of 4-graph components, where the equivalence is generated by opposite edge relation. This agrees with the term "component" for a link diagram drawn on the plane. Certainly, the number of unicursal components remains unchanged under Reidemeister moves.

Definition 3.2. A *virtual knot* is a one-component virtual link. Analogously, one defines *long virtual knots*. In this case, to define *long virtual diagrams*, instead of framed 4-graph, one has to consider the result of breaking a framed 4-graph at some edge midpoint; this will result in two vertices of valency 1. All vertices of valency 4 inherit the framing (the half-edges incident to a four-valent vertex are split into two pairs of formally opposite edges); one may also say that the newborn vertices of valency 1 are mapped to ∞ and $+\infty$ and the intersection of the image of the graph with the exterior of some large circle coincides with Ox, and the graph is oriented from $-\infty$ to $+\infty$ along its unicursal component.

Definition 3.3. A *long virtual knot* is an equivalence class of long virtual diagrams by the same Reidemeister moves. It is allowed to apply these Reidemeister moves only inside some prefixed large circle.

3.3 Virtual Knots and Gauss Diagrams

Definition 3.4. With a virtual knot K diagram one naturally associates (see [GPV]) a *Gauss diagram* $\Gamma(K)$, with all chords endowed with arrows and signs; chords of the Gauss diagrams are in one-one corresondence with *classical* crossings of the diagram; the arrow is pointed from the preimage of the overcrossing arc to the preimage of the undercrossing arc; the sign of the crossing locally looking like $\nearrow\!\!\!\!\nwarrow$ is positive, and the sign of a crossing locally looking like $\nwarrow\!\!\!\!\nearrow$ is negative. With a (long) virtual knot
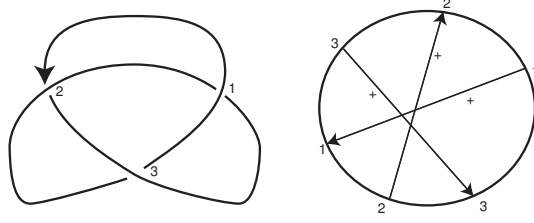
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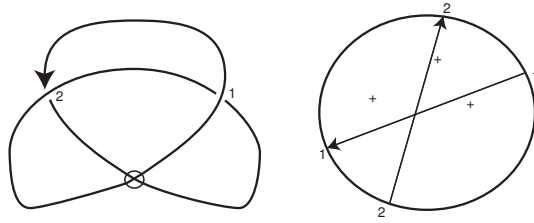Figure 6: The Right Trefoil and Its Gauss Diagram

Figure 7: The Virtual Trefoil and Its Gauss Diagram

diagram K one associates a (long) free knot whose chord diagram is obtained from $\Gamma(K)$ by forgetting arrows and signs. Note that the detour move does not change the Gauss diagram corresponding to a virtual diagram. By construction, Reidemeister moves on virtual knots generate the equivalence relations on free knots described above also called *Reidemeister moves*.

The Gauss diagram of classical and virtual trefoils are given below.

By *virtualization move* for a classical crossing we mean the local transformation of a (virtual) diagram, which inverts the arrow direction for the arrow corresponding to this crossing, and preserves the sign of the crossings (in the level of Gauss diagrams). This map is well defined up to detour moves. In the language of virtual diagrams the virtualization move looks as shown in Fig. 8.

In [Chr], Chrisman proved that Goussarov-Poylak Viro combinatorial formulae for virtual knots do not yield invariants preserved by virtualization. The invariants of virtual knots constructed in the present paper are all invariant under virtualization.
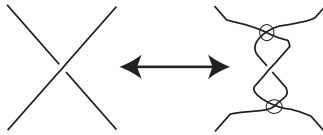

Figure 8: The virtualization move

4 Vassiliev (finite-type) invariants. The main result

Let h be an invariant of (long) virtual knots. We say that h is a *finite-type (Vassiliev)* invariant of order at most k, if for every virtual knot K with $k+1$ fixed classical crossings the following takes place $\sum_s (-1)^{\#(s)} h(K_s) = 0$, where the sum is taken over all possible 2^{k+1} knots K_s coiniciding with K everywhere except neighborhoods of given crossings, and at the fixed $k+1$ places, the crossings $\times$ or $\times$ are chosen arbitrarily. Here the symbol $\#(s)$ stays for the number of crossings of type $\times$ among the chosen ones.

The other way of defining finite-type invariants uses *singular knots* and *rigid vertices*, for more details see, [Ka1]. A rigid vertex $\times$ is a formal linear combination $\times = \times - \times$. A *singular knot* of order k is a formal linear combination of 2^k knots which appear as resolutions of k formal singular crossings (rigid vertices). Every knot invariant naturally extends to linear combinations of knots, hence, to singular knots. An invariant is of finite type at most k whenever its extension to singular knots of order $k+1$ vanishes.

Our next goal is to use a wider group than G_m to construct invariants of *virtual knots* (with some over/undercrossing information). This goal can be achieved by extending the group G_m.

Consider the group $\tilde{G}_m = \langle a_0', a_0'', \ldots, a_{m-1}', a_{m-1}'', a_m | a_m^2 = e, (a_i')^2 = (a_i'')^2, i = 0, \ldots, m - 1, a_i' a_j' = a_j' a_i'', a_i' a_j'' = a_j'' a_i'', a_j' a_i' = a_i'' a_j'', a_j'' a_i' = a_i'' a_j'', i < j; a_i' a_m = a_m a_i'', i = 0, \ldots, m - 1 \rangle$.

Let us fix a natural number m, and write $G = G_m$. It is clear that the group G defined above is obtained from $\tilde{G}_m$ by adding the following defining relations: $(a_i')^2 = 1, i = 0, \ldots, m - 1$. Let $\mathcal{G} = \mathbb{Q}\tilde{G}_m$ be the group algebra of $\mathcal{G}$, and let $\mathcal{G}_k$ denote the quotient algebra of $\mathcal{G}$ by the following relations $\prod_{j=1}^{k+1}((a_{n_j}')^2 - 1) = 0$, where n_j stays for any arbitrary set of numbers from 0 to $m - 1$. It is clear that $\mathcal{G}_0 = \mathbb{Q}G$.

From the relations for the group G, it easily follows that every element of $\mathcal{G}_k$ looking like $\alpha_1 \cdot A_1 \cdot \alpha_2 A_2 \cdots \alpha_{k+1} A_{k+1} \cdot \alpha_{k+2}$ for arbitrary α and $A_j = (u - u^{-1})$, where u stays for a generator of the group G, is *equal to zero*. Indeed, it suffices to note that the commutation relations for u and u^{-1} are similar, e.g. for $u = a_1'$ we have $a_i' u = a_1''$ and $a_i' u^{-1} = (a_1'')^{-1}$ for $i > 1$. So, $(u - u^{-1})$ in this case transforms into $a_1'' - (a_1'')^{-1}$. So, we can collect all expressions of type $(u - u^{-1})$ together, and get 0.

Let K be a long virtual knot, (see [Ma2]). With it, we associate an element $\delta(K) \in \tilde{G} \subset \mathcal{G}$ as follows. Take a Gauss diagram $\Gamma(K)$ of K, and start writing a word in generators of the group $\tilde{G}$ and its inverses exactly in the way we were writing down the word $\gamma(D)$, with the only difference that instead of each generator a_j' or $a_j'', j = 0, \ldots, m - 1$ we shall write down either this generator or its inverse depending on whether the crossing in question is positive ($\times$) or negative ($\times$). Denote the obtained word by $\delta(K)$. Note that the word $\delta(K)$ by definition does not depend on the direction of arrows in the Gauss diagram of K.

The main result of the present paper is the following

Theorem 2. *The element $\delta(K)$ of the group $\tilde{G}$ is an invariant of long virtual knots, which does not change under virtualization (change of arrow direction on the Gauss diagram).*

The conjugacy class of the element $\delta(K)$ in $\tilde{G}$ is an invariant of (compact) virtual knot.

For every k, the map $K \to \delta(K) \in \mathcal{G}_k$ is a Vassiliev invariant of long virtual knots of order less than or equal to k.
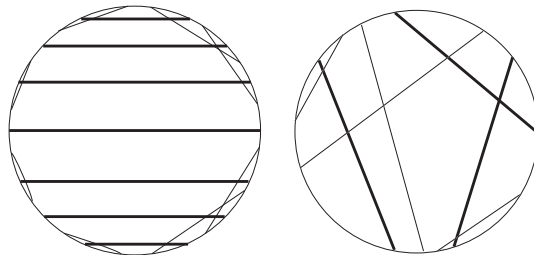
Figure 9: Two non-slice free knots

Proof. The first statement of the theorem easily follows from a comparison of Reidemeister moves for virtual knots and the relations in the group $\tilde{G}$; by construction, the invariant δ does not depend on the arrow direction on the Gauss diagram: the latter statement means the invariance under virutalization (see [FKM]). When passing from long virtual knots to compact virtual knots, we allow the marked point to go along the Gauss diagram (or, equivalently, we allow the infinity change); when moving the marked point through a crossing we conjugate the correspondent element of $\tilde{G}$, which yields the second statement of the theorem.

Now, consider the alternating difference of the values of δ for all long virtual knots corresponding to $k+1$ choices crossing types $\times$ and $\times$ (or, equivalently, the value of δ on the singular knot of order $k + 1$). By construction of δ, each singular crossing will contribute a factor of one of the following types: $(a_i' - (a_i')^{-1})$ or $(a_i'' - (a_i'')^{-1})$ or $(a_m - (a_m)^{-1})$; the latter factor is zero. Taking into account the factorization relation defining $\tilde{G}_k$, and the fact that the number of factors in our product is equal to $k + 1$, we conclude that the desired alternating sum in $\tilde{G}_k$ is equal to zero. $\square$

The invariants described in the present paper are constructive. The group $\tilde{G}_m$ admits a simple description similar to that of the group G. The Cayley graph of the group $\tilde{G}_m$ is the integer grid in $\mathbb{R}^{2n+1}$; all coordinates of the vertices of this graph are arbitrary integer numbers, except the last one, which is equal to either zero or one. The right multiplication by a_m changes the last coordinate, whence the right multiplication by a_i' or a_i'' corresponds to a shift in positive direction along one of coordinates x_{2i+1} or x_{2i+2}; here the choice which coordinate changes depends on the parity of sum of all coordinates from $2i + 1$ to m.

5   Values of the Invariants and Further Discussion

First we note that even for free knots and even for the group G_1, the values of the invariant $[\gamma]$ are very interesting. In Fig. 9, we give two knots where the two coordinates of these values are equal to 16 and 8, respectively.

The knots above are not *slice* in the sense of paper [Ma4]. In fact, the invariant $[\gamma]$ provides a sliceness obstruction for free knots, as shown in [Ma4].

It is not a difficult exercise to show that the $L(K)$ is divisible by 4 for every free knot. The divisibility of this invariant is conjectured by O.V.Manturov and will be proved elsewhere.

So, one can see that even the invariant of free knots and even for the case of the group G_1 is highly non-trivial. Certainly, so are invariants of long virtual knots.

In the present paper, we have constructed finite-type invariants of *long* virtual knots. For compact virtual knots, this can be done as well, but not in that elegant manner: one cannot further deal with specific elements of $\tilde{G}_m$, but should rather take conjugacy classes; the problem of extracting conjugacy classes from one another and taking alternating sum looks cumbersome. We shall touch on this problem in a separate publication.

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